ISRAEL JOURNAL OF MATHEMATICS, Vol. 46, Nos. 1-2, 1983

POLYHEDRAL 2-MANIFOLDS IN E^3 WITH UNUSUALLY LARGE GENUS

BY
P. McMULLEN, CH. SCHULZ AND J. M. WILLS

ABSTRACT
An equivelar polyhedral 2-manifold in the class $\mathcal{M}_{p,q}$ is one embedded in E^3 in which every face is a convex p-gon and every vertex is q-valent. In this paper, examples are constructed, to show that each of the classes $\mathcal{M}_{3,q}$ ($q \geqq 7$), $\mathcal{M}_{4,q}$ ($q \geqq 5$) and $\mathcal{M}_{p,4}$ ($p \geqq 5$) contains infinitely many distinct combinatorial types. As particular examples, there are polyhedral 2-manifolds with 576 vertices and genus 577, and with 4096 faces and genus 4097. A modification of one construction shows that there is a constant k, such that for each $g \geqq 2$, there exists a closed polyhedral 2-manifold in E^3 of genus g with at most $kg/\log g$ vertices.

§1. Introduction

A *polyhedral 2-manifold* or, more briefly (and only for the purpose of this paper), a *polyhedron* is a closed (that is, lacking boundary) topological 2-manifold M in some euclidean space E^d, which is the underlying point-set of a geometric 2-complex, in the sense of, for example, Grünbaum [5], §3.2. Thus the *faces* or 2-cells of M are convex polygons, the 0- and 1-cells of M are similarly also referred to as its *vertices* and *edges*. We shall also usually demand that adjacent faces, which share a common edge, not be parallel; from a theoretical view-point, this requirement may be unnecessary, and in any case is usually not difficult to impose in what we do below. Since we are also concerned here with such polyhedra which can be embedded in E^3, and so are necessarily oriented, we confine our attention to oriented polyhedra.

We write $f_j(M)$ for the number of j-cells of a polyhedron M for $j = 0, 1, 2$ (strictly speaking, $f_j(M)$ is the number of j-cells of the underlying cell complex, but it is usually unnecessary to make this distinction). The proof of Heawood's

Received August 1, 1982

map colouring conjecture by Ringel and Youngs [8] exhibits a certain sequence M_g ($g \geqq 0$) of polyhedra of genus g, with

$$f_0(M) =]\tfrac{1}{2}(7 + \sqrt{48g + 1})[$$

($]x[$ denotes the smallest integer $\geqq x$) for $g \neq 2$ (and $f_0(M_2) = 10$), which for simple combinatorial reasons are the minimum possible. Now for the trivial case $g = 0$, and the more complicated cases $g = 1$ solved by Császár [4] and $g = 2$ and 3 solved by Brehm [2], there are polyhedra M_g with this minimal number of vertices embeddable in E^3; in general, however, whether such polyhedra M_g are embeddable in E^3 is an open question. In fact, it is even unknown whether it is possible to achieve $f_0(M_g) = O(g^\alpha)$ for some $\alpha < 1$; hitherto, the best bound is $f_0(M) \leqq \beta g + c_\beta$, for any $\beta > \frac{3}{2}$ and some c_β, which is due to Schulz [9]. One of the main results of the present paper is to improve this bound to $f_0(M_g) = O(g/\log g)$.

The "dual" problem, that of minimizing $f_2(M_g)$, seems to be even less tractable. Here, of course, the fact that the faces have to be convex imposes a strong restriction. For example, if all vertices are 3-valent, M_g must be the boundary of an ordinary convex polyhedron (and this remains true for embeddability in any E^d; see Grünbaum [5], exercise 11.1.7). For such reasons, perhaps we should not expect results as strong as for $f_0(M_g)$; indeed, here we can only obtain $f_2(M_g) = O(g/\log g)$ for a subsequence of values of g.

On the general question of embeddability of polyhedra, let us merely remark that, from a result of Perles (compare [5], 11.1.8), it follows that if a polyhedron M is embeddable in some euclidean space E^d, then an isomorphic polyhedron (in fact an affine image of M) is embeddable in E^5. In particular, the polyhedra of Ringel and Youngs [7], which have triangular faces, are so embeddable (because each simplicial k-dimensional complex is embeddable in E^{2k+1}).

Our approach to these problems uses a concept introduced in an earlier paper (McMullen, Schulz and Wills [6], which in the future we refer to as MSW), which is also of independent interest. We say a polyhedron M is *equivelar* of type $\{p, q\}$ or in the class $\mathcal{M}_{p,q}$, if every face of M is a p-gon and every vertex of M is q-valent. We also write $M = \{p, q; g\}$ if we wish to put stress on the genus g of M. This notation, which is an obvious adaptation of that of Coxeter [3], should not be taken to imply the uniqueness of the combinatorial structure of M, and certainly not any properties such as combinatorial regularity.

In the following three sections, then, we shall prove the three parts of

THEOREM 1. *Each of the following classes $\mathcal{M}_{p,q}$ contains infinitely many different combinatorial types:*

(a) $\mathcal{M}_{3,q}$ *for* $q \geqq 7$;
(b) $\mathcal{M}_{4,q}$ *for* $q \geqq 5$;
(c) $\mathcal{M}_{p,4}$ *for* $p \geqq 5$.

By a modification of the construction which yields part (a), we shall also show in the last section

THEOREM 2. *There is a constant k, such that, for each* $g \geqq 2$, *there exists a polyhedron* M_g *of genus g, with*

$$f_0(M_g) \leqq kg/\log g.$$

§2. The construction of $\{3, q\}$

Before describing our construction of the polyhedron $\{3, q\}$ geometrically, we give a purely combinatorial description of a rather more general idea.

Let M be a (combinatorial) polyhedron, and let $F_1, \cdots, F_n$ be n disjoint faces of M. If, for $i = 0, 1, M^i$ is an isomorphic copy of M, with corresponding faces $F_1^i, \cdots, F_n^i$, then we can *delete* the faces F_j^0 of M^0 and F_j^1 of M^1, and for each j, join the boundaries of F_j^0 and F_j^1 by a *tube* of quadrangles. We then obtain a new (combinatorial) manifold N, with

$$f_0(N) = 2f_0(M), \qquad g(N) = 2g(M) + n - 1.$$

The quadrangles of each tube can be split into pairs of triangles — in many different ways, of course — but in exactly two ways will there be three new triangles meeting each original vertex of M_0 (and of M_1).

In our applications, M will be a polyhedron $\{3, q\}$, and the triangles F_j will cover the vertices of M (exactly once), so that $n = \frac{1}{3}f_0(M)$. It is then clear that, with either of the two special choices of splitting the quadrangles of the tubes, N will be of type $\{3, q + 2\}$. We refer to this procedure as *method D* (compare MSW, §4).

The combinatorial idea behind our construction is thus very simple. The difficulty arises when we try to realize the construction geometrically in E^3. Our method will be inductive in two senses: firstly (and more easily) M must have such a set of disjoint faces covering its vertices; secondly, M must be geometrically suitable, in a sense which we now make precise. In fact (and herein lies the basic trick), our M will not strictly speaking be a polyhedron of type $\{3, q\}$ at all, since certain of its adjacent faces will be coplanar. This trivial disadvantage we shall overcome at the end of our constructive procedure.

So, we call a polyhedron M of type $\{3, q\}$ *special* if M itself is obtained from a polyhedron L of type $\{3, q-2\}$ by method D, in such a way that the following three conditions hold.

(1) *The copies* L^0 *and* L^1 *of* L *satisfy* $L^0 \cap L^1 = \varnothing$.

We can, in fact, assume that $L^1 \subseteq I(L^0)$, the *interior* (that is, bounded) component of $E^3 \setminus L^0$; however, the relationship between L^0 and L^1 is symmetrical.

For each cell F of L, with corresponding cells F^i of L^i $(i = 0, 1)$, we write $Q = \operatorname{conv}(F^0 \cup F^1)$; similarly, Q_j is obtained from the cell F_j of L. Then

(2) *If* F, F_1, F_2 *are cells of* L *with* $F_1 \cap F_2 = F$, *then* $Q_1 \cap Q_2 = Q$.

The final condition concerns the deleted faces of L. If $F = abc$ is a deleted face of L, we write $F^i = a^i b^i c^i$, and so on. Then

(3) *The tube based on each deleted face abc of* L *satisfies*:

(a) *the pairs of edges* a^0b^0, a^1b^1 *and* a^0c^0, a^1c^1 *are parallel*;

(b) *the interior component* $I(M)$ *is convex at the edge* b^0c^1, *that is, the non-edge* b^1c^0 *lies in* $\operatorname{cl} I(M)$.

Note the conditions (3), and the fact that M is obtained by method D, imply that a^0b^1 and a^1c^0 are edges of M, and that the pairs of faces $a^0b^0b^1$, $a^0a^1b^1$ and $a^0a^1c^0$, $a^1c^0c^1$ are coplanar. Conditions (1) and (2) are clearly independent, and together have the following implication.

Lemma 1. *If* L^0 *and* L^1 *satisfy* (1) *and* (2) *above, then for each cell* F *of* L, $Q \cap L^i = F^i$ *for* $i = 0, 1$.

We first observe that we can confine our attention to faces F. For, a general cell F can be expressed as $F = \bigcap F_j$, where the F_j are faces; if the lemma holds for faces, then

$$Q \cap L^0 = \left(\bigcap Q_j\right) \cap L^0 = \bigcap (Q_j \cap L^0) = \bigcap F_j^0 = F^0,$$

as required.

So, suppose that F is a face, such that $Q \cap L^0 \neq F^0$, so that $(Q \setminus F^0) \cap L^0 \neq \varnothing$. Hence there is a face $F_1 \neq F$, such that $(Q \setminus F^0) \cap F_1^0 \neq \varnothing$. Now, $F \cap F_1 = F_2 \neq \varnothing$, otherwise $Q \cap F_1^0 \subseteq Q \cap Q_1 = Q_2 = \varnothing$. Since, by (1), $F_1^0 \cap L^1 \subseteq L^0 \cap L^1 = \varnothing$, F_1^0 has some edge $F_3^0 \neq F_2^0$, such that $(Q \setminus F^0) \cap F_3^0 \neq \varnothing$. Now $F \cap F_3 = F_4 \neq \varnothing$, otherwise $Q \cap F_3^0 \subseteq Q \cap Q_3 = Q_4 = \varnothing$, so that F_4 is a vertex, and Q_4 is a line segment. From

$$\emptyset \neq (Q \setminus F^0) \cap F_3^0 = (Q \cap F_3^0) \setminus (F^0 \cap F_3^0) = (Q \cap F_3^0) \setminus F_4^0 \subseteq Q_4 \setminus F_4^0,$$

and (again by (1)) $F_3^0 \cap L^1 = \emptyset$, follows $F_3^0 \subseteq Q_4$. Hence the other vertex F_5^0, say, of the edge F_3^0 lies in Q_4. But $F_4 \cap F_5 = \emptyset$, so that $F_5^0 \subseteq Q_4 \cap F_5^0 \subseteq Q_4 \cap Q_5 = \emptyset$, and thus we have obtained our required contradiction. This proves Lemma 1.

This result can be interpreted as saying that if L^0 and L^1 satisfy (1) and (2), then each Q lies between L^0 and L^1. More precisely, if $E(L^1)$ is the *exterior* (unbounded) component of $E^d \setminus L^1$, then $Q \subseteq \mathrm{cl}(I(L^0) \cap E(L^1))$.

Moreover, for each cell F of L, Q has opposite faces (in the sense of convex sets) F^0 and F^1. If $\dim F = 1$, then Q is a quadrangle or tetrahedron; otherwise, $\dim Q = \dim F + 1$. In case $\dim F = 1$ or 2, the vertices of F also give rise to edges of Q, joining the corresponding vertices of F^0 and F^1.

Before our next Lemma, we introduce some notation. Let L^0, L^1 satisfy (1) and (2). If a is a vertex of L, for $0 < \lambda < 1$ we write $a^\lambda = (1-\lambda)a^0 + \lambda a^1$. If the face F of L has vertices a, b and c, we write $F^\lambda = \mathrm{conv}\{a^\lambda, b^\lambda, c^\lambda\}$. Finally, we write L^λ for the union of these triangles F^λ.

Lemma 2. *L^λ, as defined above, is a polyhedron naturally isomorphic to L, and, with L^λ replacing L^1, L^0 and L^λ satisfy conditions* (1) *and* (2).

For the first part, we must show that if F, F_1 and F_2 are cells of L with $F_1 \cap F_2 = F$, then $F_1^\lambda \cap F_2^\lambda = F^\lambda$. Certainly, we have $F_1^\lambda \cap F_2^\lambda \supseteq F^\lambda$. If $F = \emptyset$, then $F_1^\lambda \cap F_2^\lambda \subseteq Q_1 \cap Q_2 = Q = \emptyset$; so we have equality in this case. Otherwise, we can check the result case by case (according to the dimensions of the cells); the only case that can cause problems is when F_1 and F_2 are faces sharing a common edge F, and this leads to disjoint cells $F_3 \subseteq F_1$ and $F_4 \subseteq F_2$ with $F_3^\lambda \cap F_4^\lambda \neq \emptyset$, which is already excluded.

Next, for condition (1), suppose $L^0 \cap L^\lambda \neq \emptyset$. Then there are cells F_0, F_1 of L, with $F_0^0 \cap F_1^\lambda \neq \emptyset$. But $F_0^0 \cap F_1^\lambda \subseteq L^0 \cap Q_1 = F_1^0$ by Lemma 1, and hence $F_1^0 \cap F_1^\lambda \neq \emptyset$. But this is clearly impossible (compare the remarks after Lemma 1).

Finally, we check condition (2). If F, F_1, F_2 are cells of L with $F_1 \cap F_2 = F$, then, with $Q^\lambda = \mathrm{conv}(F^0 \cup F^\lambda)$, and so on, we have $Q^\lambda \subseteq Q_1^\lambda \cap Q_2^\lambda \subseteq Q_1 \cap Q_2 = Q$. The cases $\dim F \leqq 0$ are thus straightforward; so, the only possible problem can again occur only when F_1 and F_2 are faces sharing a common edge F, and this is an elementary problem in solid geometry, which we leave to the interested reader. (Again we refer the reader to the remarks after Lemma 1.)

Lemma 3. *Let $0 < \lambda < \mu < 1$, and let M^* be constructed from L^λ and L^μ as M is from L^0 and L^1, with a^λ replacing a^0 and a^μ replacing a^1 for each vertex a of*

L. Then M^* is a special polyhedron naturally isomorphic to M, $M^* \subseteq \operatorname{cl} I(M)$, and $M \cap M^*$ is contained in the union of the tubes of M.

That M^* satisfies (1) and (2) follows by a double application of Lemma 2 (to L^0 and L^μ, then to L^λ and L^μ). For condition (3a) (see Fig. 2.1), note that $a^\lambda b^\lambda$, $a^\mu b^\mu$ are both parallel to a^0b^0, a^1b^1, and hence to each other; similarly, $a^\lambda c^\lambda$, $a^\mu c^\mu$ are parallel. Condition (3b) is also clear (consideration of the intermediate case M^μ obtained from L^0 and L^μ makes it transparent). Finally, $M^* \subseteq \operatorname{cl} I(M)$ by the convexity condition (3b) for M, and $M^* \cap (L^0 \cup L^1) = \emptyset$ by Lemmas 1 and 2, which has the required implication about $M \cap M^*$.

Our final step is slightly to modify M^*. We begin with a more general result.

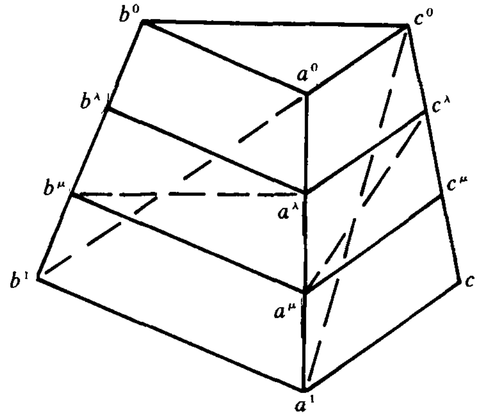


Fig. 2.1.

LEMMA 4. *Properties* (1) *and* (2) *are preserved under all sufficiently small perturbations of the vertices of M.*

This is clear for perturbations of one vertex, and the general case follows from the finiteness of M.

Our specific modification of M^* is the following. In the tube of M based on the triangle abc of L, we pick a point $p \in \operatorname{conv}\{a^0, b^0, c^0, a^1, b^1, c^1\} \cap E(M)$. From the points a^λ, a^μ, we obtain new points

$$\bar{a}^\lambda = (1-\nu)p + \nu a^\lambda,$$

and similarly for $\bar{a}^\mu$, where $\nu > 1$.

LEMMA 5. *If ν is sufficiently near* 1, *then the points $\bar{a}^\lambda$, $\bar{a}^\mu$ are vertices of a special polyhedron $\bar{M}^*$ naturally isomorphic to M, with $\bar{M}^* \subseteq I(M)$.*

This follows from Lemma 4, and the remark that the local homotheties $a^\lambda \to \bar{a}^\lambda$ ($a^\mu \to \bar{a}^\mu$) preserve the parallelism of (3a) and the properties (3b). Clearly, these local homotheties move M^* into $I(M)$. This situation is illustrated in Fig. 2.2.

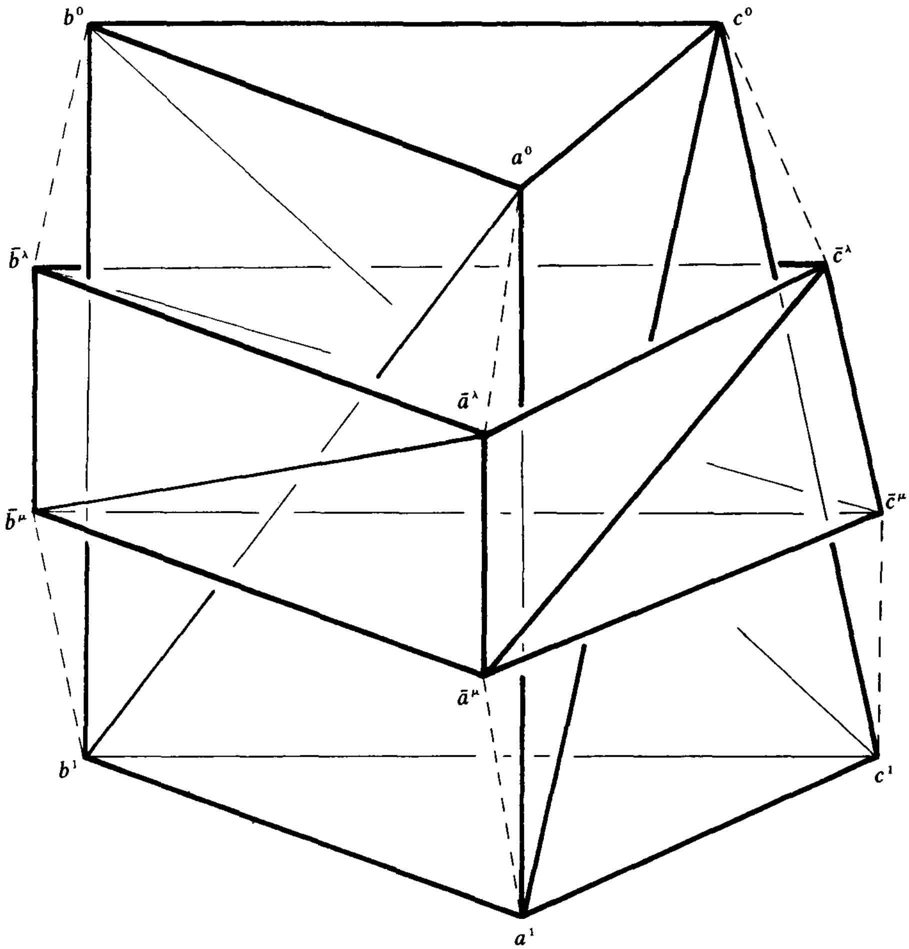


Fig. 2.2.

The inductive step is completed by

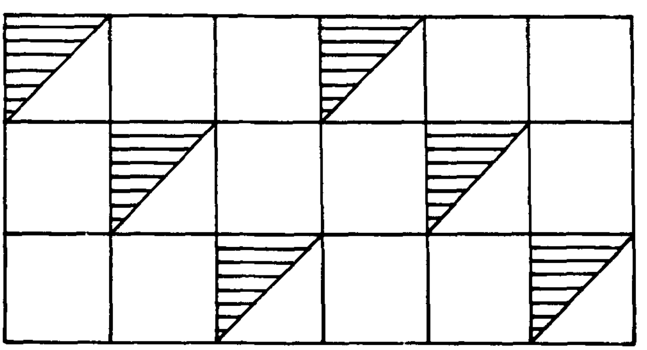
LEMMA 6. *With $\bar{M}^*$ as above, let $M^0 = M$, $M^1 = \bar{M}^*$, and let N be obtained from M by method D, with deleted triangles from the tube based on abc being $b^0b^1a^0$ and $c^1a^1c^0$ (with b^0, c^1 corresponding to a, and so on). Then N is a special polyhedron.*

Conditions (1) and (3) are obvious (note that the local homotheties of Lemma 5 preserve the required parallelisms for (3a), and the ordering of the vertices of the deleted triangles ensures (3b)). So, it remains to check (2). Now, to the vertices a^0 and a^1 of $M = M^0$ correspond the vertices $\bar{a}^\lambda$ and $\bar{a}^\mu$ of $\bar{M}^* = M^1$, respectively. For the faces F^0 or F^1 of M which were originally faces of L^0 or L^1, (2) is easy; either the Q's both come from L^0, and the condition follows from Lemma 1 (and similarly for the Q's coming from L^1), or one comes from L^0 and one from L^1, and the condition is trivial, since (say) Q_1 lies between L^0 and $\bar{L}^\lambda$ and Q_2 lies between L^1 and $\bar{L}^\mu$. The remaining cases involve cells, at least one of which lies in a tube; these cases follow directly, by inspection (note that two different tubes of M cannot interfere with each other, so the checking is purely local). We refer the reader once more to Fig. 2.2.

We have now established the inductive step; it therefore remains to give the initial examples.

LEMMA 7. *There are infinitely many combinatorially distinct special polyhedra in* $\mathcal{M}_{3,8}$.

We must construct suitable pairs L^0, L^1 in $\mathcal{M}_{3,6}$. Let $m, n \geqq 1$. Take two concentric and homothetic regular $3m$-gons in a plane in E^3, and rotate this plane by angles $2k\pi/3n$ $(k = 0, 1, \cdots, 3n-1)$ about a line in this plane which misses the polygons. This yields two tori, one inside the other, of which corresponding edges and faces are parallel. The faces of these tori are quadrangles, and we divide each quadrangle into two triangles, as in Fig. 2.3, to yield a $\{3, 6\}$.



$m = 2 \qquad n = 1$

Fig. 2.3.

The deleted triangles are also depicted in Fig. 2.3; two sides of each are parallel to the corresponding sides of corresponding triangles of the other torus, and obviously a labelling can be chosen to satisfy the conditions of (3).

The final example is a little more complicated.

LEMMA 8. *There are infinitely many combinatorially distinct special polyhedra in* $\mathcal{M}_{3,9}$.

We must first construct the suitable examples in $\mathcal{M}_{3,7}$. Now there are infinitely many combinatorially distinct simple 3-polytopes P, all of whose faces are triangles or hexagons (in fact, all faces multi-3-gons would suffice; see [5], chapter 13). For such a 3-polytope P, let $S(P)$ be its snub polytope (see [3] for the precise definition). To each face of P corresponds a face of $S(P)$ of the same kind (triangle or hexagon); each vertex of such a face belongs to four other triangles, so the vertex is 5-valent; the faces corresponding to the original faces of P are disjoint, and cover the vertices of $S(P)$. (For further details, see MSW, §4.)

Let $L^0 = \text{bd}\, S(P)$, and from an interior point p of $S(P)$, let L^1 be a homothetic copy $L^1 = (1-\nu)p + \nu L^1$, with $0 < \nu < 1$. Now delete the faces of L^0 and L^1 corresponding to the original faces of P, and join corresponding boundaries by tubes of quadrangles. Then split each quadrangle into two triangles, as in Fig. 2.4.

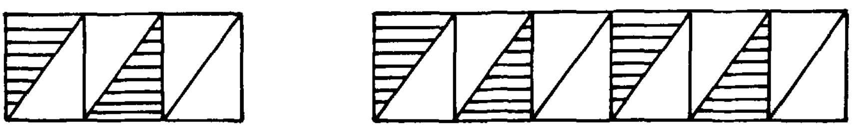


Fig. 2.4.

The resulting polyhedron M is not quite special, but only because some of the tubes are based on hexagons. But it is obvious that the method of the inductive construction carries over to this case, using the deleted faces indicated in Fig. 2.4. So, we eventually obtain infinitely many special polyhedra of type $\{3, 9\}$ (since M is of type $\{3, 7\}$).

This discussion completes the proof of Theorem 1a, when we add the remark that moving the vertices into general position at the end destroys any incidental coplanarities of adjacent faces.

A suitable choice of a $\{3, 6\}$ (as described above) with 9 vertices enables us to prove a slightly stronger result.

LEMMA 9. *For each $r \geqq 3$, there is an $M \in \mathcal{M}_{3,2r}$ with $9 \cdot 2^{r-3}$ vertices, two faces of which are faces of* conv M.

However we choose the triangle of the torus of Lemma 5 in case $m = n = 1$ to be deleted, two faces remain on the boundary of the convex hull of the ultimate polyhedron M. Suitably moving the vertices of M into general position preserves this property.

We may also note that the case $r = 9$ of Lemma 9 gives a polyhedron $\{3, 18\}$ with 576 vertices and genus 577 (compare MSW, §2); this is the minimal example of which we are aware with more "holes" than vertices.

§3. The construction of $\{4, q\}$

The construction we describe in this section for the polyhedra $\{4, q\}$ is quite different from that for $\{3, q\}$. (There is, in fact, an analogous construction which, in contrast to $\{3, q\}$, has some symmetries, but it is somewhat longer, and we therefore omit it.) Here, we begin by building up what we shall call a corner of the polyhedron by an inductive method, and only at the end do we fit these corners together (compare Figs. 3.1–3.4).

At the q-th stage ($q \geqq 4$), a *corner* will have 2^{q-4} vertices, each of which lies in $q - 2$ faces (which may be finite quadrilaterals, or infinite half-strips or quarter-planes) and $q - 1$ edges. Of these $q - 1$ edges, $q - 3$ will be *tied*, that is, shared by two faces, and the other 2, both of which are half-lines, are *free*. The corner will lie in the non-negative orthant, and one set of free edges (which we call *horizontal*) will be parallel to the x-axis, while the other (*vertical* edges) will be parallel to the z-axis. Through each vertex will also pass a tied edge, also a half-line, which is parallel to the y-axis. In addition, all the free edges E will be *visible* from the direction $(0, -1, 0)$, which means that the infinite quarter-planes or half-strips

$$E + \{\lambda(0, -1, 0) \mid \lambda \geqq 0\}$$

meet each other and the faces of the corner in common edges at most. Initially (in case $q = 4$), we have two quarter-planes meeting at a common edge.

Our construction proceeds as follows (Figs. 3.1–3.4, free edges with heavy lines). We choose a plane H perpendicular to the x-axis which cuts all the horizontal infinite edges, and has all the vertices of the corner strictly to one side of it. Truncate the corner by H, discarding the part containing no vertices, and adjoin to the truncated corner its reflected copy in H. So, the number of vertices has doubled. We can now choose α, $\beta > 0$, such that the new infinite faces

$$E + \{\lambda(\alpha, -1, -\beta) \mid \lambda \geqq 0\},$$

where E is a free horizontal edge (which is now finite), are such that their infinite (free) edges and the vertical free edges (old and new) are all visible from $(0, -1, 0)$. (First take $\alpha = 0$ and choose β, then increase α.) We now apply a shear, which takes the y- and z-axes into themselves, and makes $(\alpha, -1, -\beta)$ parallel to the x-axis. This preserves visibility from $(0, -1, 0)$, and gives us our new corner, increasing q by 1.

We fit the corners together as follows. Take the corner at stage q. Let $m, n \geqq 2$.

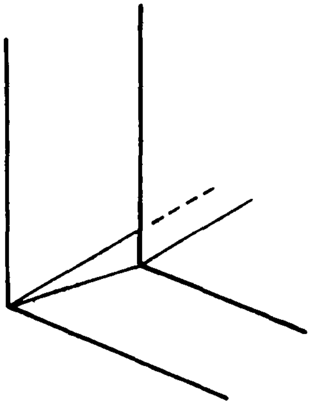


Fig. 3.1 ($q = 5$).

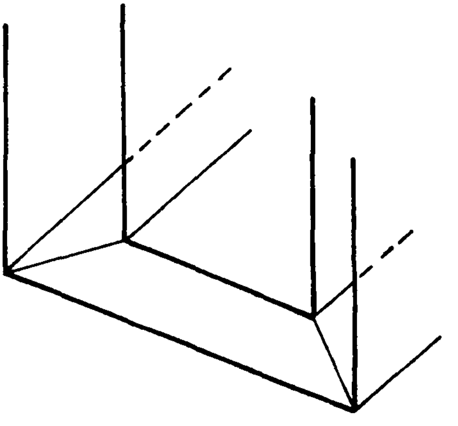


Fig. 3.2.

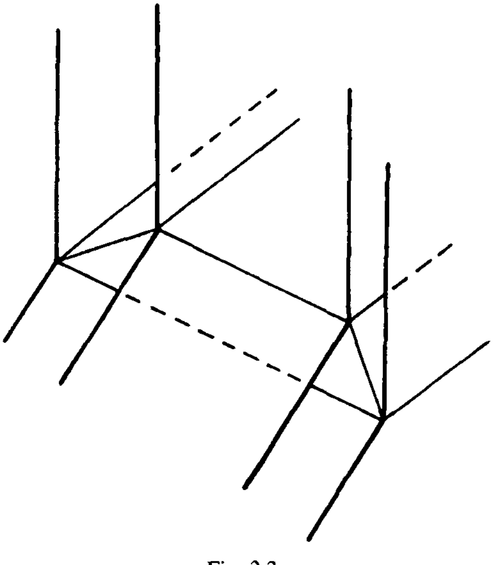


Fig. 3.3.

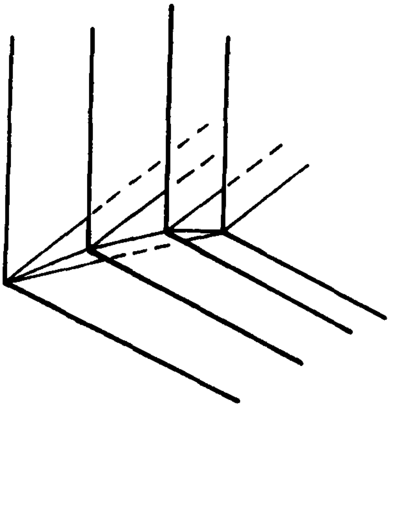


Fig. 3.4.

Perform a shear taking the y- and z-axes into themselves, and taking the x-axis into the xy-plane, with an angle $\pi - \pi/m$ between it and the z-axis. Truncate the corner by planes perpendicular to the free edges (as before, the vertices all lie to one side); the angle between these planes is π/m. We now fasten together the $2m$ copies obtained by repeated reflexions in these planes.

Now, all the free edges (which form $2m$-gons) are visible from some direction making a (small) non-zero angle with $(0,-1,0)$. Perform a shear making the angle between this direction and the y-axis $\pi - \pi/n$; say the new direction is e. Adjoin all the faces

$$E + \{\lambda e \mid \lambda \geqq 0\},$$

where E is a free edge, and truncate by planes perpendicular to the two sets of infinite edges; the angle between these planes is π/n. Finally, we fasten together the $2n$ copies obtained by repeated reflexions in these planes.

We easily check that the resulting manifold is of type $\{4, q\}$ with $2^{q-2}mn$ vertices, and hence with genus $(q-4)2^{q-5}mn + 1$. Figure 3.5 shows the case $q = 5$; $m = 2$, $n = 3$.

Manifolds of type $\{4, q\}$ isomorphic to those constructed above for $m = n = 2$ are contained in the 2-skeleton of the q-cube (see Ringel [7]).

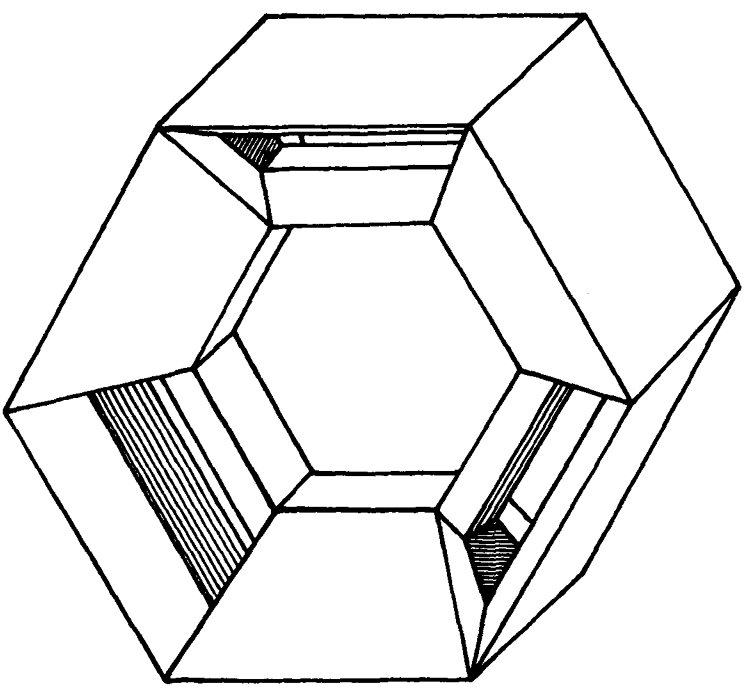


Fig. 3.5. $\{4,5;7\}$.

§4. The construction of $\{p, 4\}$

Our construction for the polyhedra $\{p, 4\}$ is quite similar in spirit to that for $\{4, q\}$. We first build a *block* of 2^{p-4} p-gons, which are attached along certain of their edges, and only at the end do we fasten copies of this block together to form the polyhedron.

Let C be the infinite cylinder in E^3 bounded by the four planes

$$x = 0, \quad x = 1, \quad y = 0, \quad y = 1.$$

Each polygon in the block will have four consecutive sides lying in the faces of C, and thus also three consecutive vertices (all four if $p = 4$, of course). We suppose these vertices to have coordinates (x, y, z) satisfying

$$(x, y) = (0, 1), \quad (0, 0) \quad \text{and} \quad (1, 0).$$

We say a vertex of a polygon in the block is *of type* k if it lies in k polygons of the block. The three vertices just mentioned will be of type 1 (all four if $p = 4$), the two remaining vertices of each polygon lying in the faces $x = 1$ and $y = 1$ will be of type 2 (for $p \geqq 5$), and all the rest of the vertices will be of type 4 (for $p \geqq 6$). Each of these last vertices is *complete*, in the sense that the four polygons fit together in a circuit around the vertex, so that all edges through the vertex are *tied* (belonging to two polygons). Through the remaining vertices pass two *free* edges, each belonging to only one polygon. The situation we have just described is illustrated in Fig. 4.1, which is a view from above of a typical polygon of the block; the label attached to each vertex is its type.

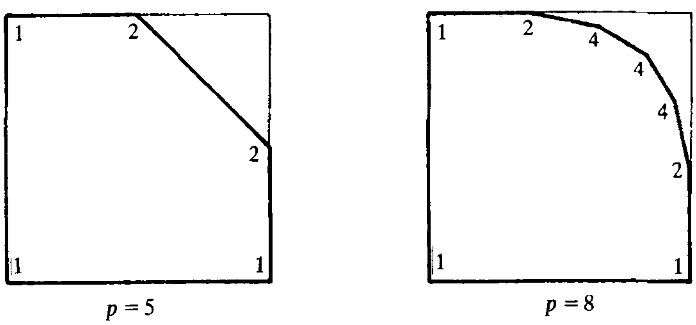


Fig. 4.1.

We can now describe the inductive procedure we use. Let us suppose that $p \geqq 5$, and that we have already constructed the block of 2^{p-5} $(p-1)$-gons. Since there is a plane $x = \lambda$ which strictly separates the vertices of the block in $x = 1$ from the remaining vertices, we see that we can perform a shear (of the form

$(x, y, z) \to (x, y, z - \mu x)$ for some μ), so that all the vertices in $x = 1$ lie strictly below all the rest (see Fig. 4.2). Translating in the z-direction, we can suppose that $z = 0$ strictly separates these two sets of vertices. We next truncate the block (and each $(p-1)$-gon) by $z = 0$ (discarding that part in $z < 0$), and take the union of the truncated block with its reflected copy in $z = 0$ (see Fig. 4.3). We

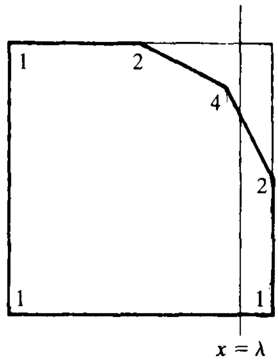


Fig. 4.2.

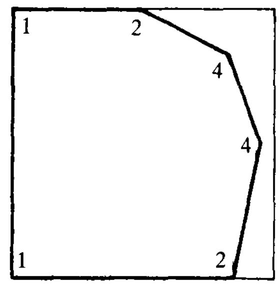


Fig. 4.3.

now have a block of 2^{p-4} $(p-1)$-gons. To the vertices of type 1 and 2 that lay in $x = 1$ now correspond vertices of type 2 and 4, respectively; the types of the remaining vertices are unchanged. Now we can find a vertical plane, of the form $x = \alpha y + \beta$ $(\alpha, \beta > 0)$, which strictly separates all the new vertices of type 2 from all the remaining vertices (the other vertices in $y = 0$ are those of type 1 lying in the line $x = 0 = y$). We next truncate the block by this plane, creating a block of 2^{p-4} p-gons. In each polygon, the vertex of type 2 is replaced by two adjacent vertices, of types 1 and 2. We complete the inductive stage of the argument by performing a suitable projective transformation of E^3 which takes the planes $x = 0$, $y = 0$ and $y = 1$ into themselves, and $x = \alpha y + \beta$ into $x = 1$; such a transformation is

$$(x, y, z) \to \frac{1}{\alpha y + \beta}(x, (\alpha + \beta)y, z).$$

These last steps are illustrated in Fig. 4.4. We may observe that a modification of this construction (using truncated half-strips or truncated half-planes, which then have to be truncated at the end to produce polygons) would enable the employment of projective transformations to be avoided. However, the construction is then possibly less intuitive.

We now fit copies of our block together to form $\{p, 4\}$. Let $m, n \geqq 2$ be arbitrary integers. As in the previous construction, we first perform a suitable shear, but now we suppose that our truncating plane is $z = (\cot(\pi/m))x$ $(z = 0,$

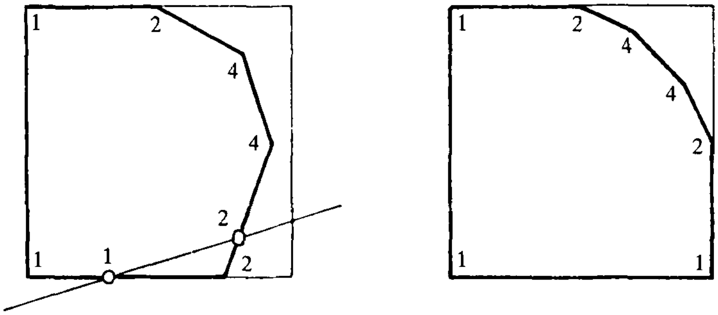


Fig. 4.4.

as we had originally, is just the case $m = 2$). We then fasten together the $2m$ copies of the truncated block, obtained by repeated reflexions in the planes $x = 0$ and $z = (\cot(\pi/m))x$, to obtain a ring. The only incomplete vertices now lie in the planes $y = 0$ and $y = 1$, and are all of type 2.

For our last step, we perform a projective transformation of E^3 taking $y = 0$ into itself and $y = 1$ into $x = (\cot(\pi/n))y$. (Alternatively, we truncate the ring by a plane slightly perturbed from $y = 1$, but not parallel to it, and then perform a suitable affine transformation.) We then fasten together the $2n$ copies of the ring, obtained by repeated reflexions in $y = 0$ and $x = (\cot(\pi/n))y$. All vertices are now complete, and are of type 4.

Thus we have a polyhedron $M = \{p, 4\}$, with

$$f_2(M) = 2m \cdot 2n \cdot 2^{p-4} = 2^{p-2}mn.$$

Hence, either directly, or by reference to MSW, we have

$$f_0(M) = p \cdot 2^{p-4}mn, \qquad f_1(M) = p \cdot 2^{p-3}mn,$$

and so

$$g(M) = (p-4)2^{p-5}mn + 1.$$

In particular, we observe that, if $p \geqq 12$, then

$$g(M) > f_2(M),$$

so that M has more "holes" than faces. In particular, we have an example of a $\{4, 12\}$ with 4096 faces and genus 4097; we know of no such example with fewer faces.

We may also observe that this construction answers a question of Barnette [1], showing that there are polyhedra in E^3, whose faces all have arbitrarily many sides.

We mention finally that in MSW fig. 6 shows the above construction for the simple case $\{5,4;5\}$.

§5. Polyhedral manifolds with few vertices

In Section 2, we constructed sequences of polyhedra $\{3,q\}$ in E^3, and, in particular, we found the minimal examples $\{3,2r\}$ ($r \geqq 3$), with

$$f_0 = 9 \cdot 2^{r-3}, \qquad g = 3(r-3)2^{r-4}+1,$$

and hence with $f_0 = O(g/\log g)$ for this special sequence of values of g. To prove Theorem 2, we must obtain such an estimate for all g.

Let us write $f_0(r)$, $g(r)$ for these values above.

Theorem 2 clearly follows immediately from

LEMMA 10. *For each $r \geqq 3$ and each $0 \leqq g \leqq g(r)$, there is a polyhedron M with $g(M) = g$ and $f_0(M) \leqq f_0(r)$.*

We may clearly make the inductive assumption, which is true for $r = 3$, that the lemma holds for $0 \leqq g \leqq g(r-1)$, since

$$f_0(r-1) = \tfrac{1}{2} f_0(r)$$

for $r > 3$.

For larger g, we proceed as follows. The polyhedron $M_r = \{3, 2r; g(r)\}$ is constructed by joining across $n_r = \frac{1}{3} f_0(r-1)$ tubes from one copy of M_{r-1} to another, so that

$$g(r) = 2g(r-1) + n_r - 1.$$

Now the construction clearly allows us to join across any subset of $n \leqq n_r$ of these tubes; that is, in $n_r - n$ cases, we have the original faces F, F' which would otherwise have been deleted. This clearly gives us any g, with

$$g = 2g(r-1) + n - 1,$$

where $1 \leqq n \leqq n_r$, and so in the range

$$2g(r-1) \leqq g \leqq g(r).$$

For the remaining cases $g(r-1) < g < 2g(r-1)$, write

$$g' = g - g(r-1).$$

By our inductive hypothesis, we can find a polyhedron M', with $g(M') = g'$ and $f_0(M') \leqq f_0(r-1)$. Moreover, the above argument and Lemma 9 show that we can assume that M' has two faces which are also faces of $\operatorname{conv} M'$. The same being true of our M_{r-1}, we can now use a familiar argument (see Grünbaum [5], exercise 5.2.17), to find a projective image M'' of M', such that M_{r-1} and M'' have one of these two faces of each in common, while

$$\operatorname{conv}(M_{r-1} \cup M'') = \operatorname{conv} M_{r-1} \cup \operatorname{conv} M'',$$

the two 3-polytopes having disjoint interiors. We now delete the shared face, giving a manifold M with

$$g(M) = g(M_{r-1}) + g(M') = g,$$

$$f_0(M) = f_0(r-1) + f_0(M') - 3 < f_0(r),$$

as we wished to show.

The argument we have given above can be refined somewhat. It is clear from the above discussion that any $g \geqq 1$ can be expressed in the form

$$g = g_1 + g_2 + \cdots + g_k,$$

where for each j, there is an r_j with

$$g_j = g(r_j) \quad (1 \leqq j < k), \qquad 2g(r_k - 1) < g_k \leqq g(r_k),$$

and

$$r_1 > r_2 > \cdots > r_{k-1} \geqq r_k \geqq 3.$$

Then there is a (triangulated) polyhedron M with $g(M) = g$ and

$$f_0(M) \leqq f_0(r_1) + \cdots + f_0(r_k) - 3(k-1).$$

Now for $r_1 = 9$, we have $g_1 = g(r_1) = 577$ and $f_0(r_1) = 576$. Thus with $k = 2$ and $r_1 = r_2$ (and so $f_0(r_2) = 576$ also), the range of g_2 above is

$$482 = 2g(8) < g_2 \leqq g_1 = 577.$$

Hence for each $g = g_1 + g_2$ in the interval $1059 < g \leqq 1154$, we have a polyhedron M with $g(M) = g$ and $f_0(M) = 577 + 577 - 3 = 1149$. Easy considerations now yield

THEOREM 3. *For each $g \geqq 1150$ (at least), there is a polyhedron M with $g(M) = g$ and $f_0(M) < g$.*

REFERENCES

1. D. W. Barnette, *Polyhedral maps on 2-manifolds*, Proceedings of the Geometry Conference in Oklahoma, 1980.

2. U. Brehm, *Polyeder mit zehn Ecken vom Geschlecht drei*, Geom. Dedic. **11** (1981), 119–124.

3. H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, 1973.

4. A. Császár, *A polyhedron without diagonals*, Acta Sci. Math. Szeged **13** (1949), 140–142.

5. B. Grünbaum, *Convex Polytopes*, Wiley-Interscience, London–New York–Sydney, 1967.

6. P. McMullen, Ch. Schulz and J. M. Wills, *Equivelar polyhedral manifolds in* E^3, Isr. J. Math. **41** (1982), 331–346.

7. G. Ringel, *Über drei Probleme am n-dimensionalen Würfel und Würfelgitter*, Abh. Math. Sem. Univ. Hamburg **20** (1955), 10–19.

8. G. Ringel and J. W. T. Youngs, *Lösung des Problems der Nachbargebiete auf orientierbaren Flächen*, Arch. Math. **20** (1969), 190–201.

9. Ch. Schulz, *Geometrische Realisierung geschlossener Flächen mit wenigen Ecken*, Geom. Dedic. **11** (1981), 309–314.

UNIVERSITY COLLEGE LONDON
LONDON, UK

FERNUNIVERSITÄT HAGEN
HAGEN, FRG

UNIVERSITÄT SIEGEN
SIEGEN, FRG